Sequences -ALMO 2025-

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Notes:

- The structure i followed focuses on getting techniques rather than seeing theory . In other words we will not study sequences analytically but we will learn how to handle weird ones through olympiad problems !
- For practice problems you can find the solution in AOPS as i gave you the source , ideally try in each problem 2 hours before spoiling the solution they are hard tho so its okay

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Chapter 1 : Basic Concepts

1.1 What is a Sequence?

A **sequence** is an ordered list of elements, usually defined by a rule that determines each term based on its position in the list. Formally, a sequence is a function

$$a : \mathbb{N} \to \mathbb{J}$$

that assigns to each positive integer n an element $a_n \in \mathbb{J}$, called the **n-th term** of the sequence.

Note: The codomain $\mathbb{J}$ can be any set—such as numbers, points, letters, strings, or even objects—depending on the problem context. However, in most Olympiad problems, unless otherwise stated (especially in combinatorics), sequences typically consist of real numbers , rationals or integers.

Notation

A sequence is often denoted by:

$$\{a_n\}_{n=1}^{\infty} \quad \text{or simply} \quad a_1, a_2, a_3, \ldots$$

In some problems, indices may start from 0, or even a general n_0, but $n = 1$ is standard.

Types of Sequences

- **Finite Sequence:** Contains a fixed number of terms.

$$\text{Example: } 2, 4, 6, 8$$

- **Infinite Sequence:** Continues indefinitely, often indexed by all natural numbers.

$$\text{Example: } 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

- **Explicit Sequence:** Each term is given directly by a formula in terms of n.

$$a_n = 2n - 1 \quad \Rightarrow \quad 1, 3, 5, 7, \ldots$$

- **Recursive Sequence:** Terms are defined based on one or more preceding terms.

$$\text{Fibonacci sequence: } a_1 = 1,\ a_2 = 1,\ a_n = a_{n-1} + a_{n-2} \text{ for } n \geq 3$$

Remarks

- Sequences are central objects in many areas of mathematics, especially in number theory, combinatorics, and recurrence-based Olympiad problems.
- Recognizing patterns and understanding the behavior of sequences is often key to solving problems involving inequalities, invariants, and recursive structures.
- It's important to distinguish between sequences as functions (with well-defined rules) and unordered collections or sets.

1.2 Monotonicity of a Sequence

A sequence is said to be **monotonic** if its terms consistently increase or decrease as the index increases.

Types of Monotonicity

Let $\{a_n\}$ be a sequence. Then:

- **Increasing:** $a_{n+1} \geq a_n$ for all n
- **Strictly Increasing:** $a_{n+1} > a_n$ for all n
- **Decreasing:** $a_{n+1} \leq a_n$ for all n
- **Strictly Decreasing:** $a_{n+1} < a_n$ for all n
- **Non-monotonic:** A sequence that does not follow any of the above patterns globally

Examples

- $a_n = n^2$ is strictly increasing.
- $a_n = (-1)^n$ is non-monotonic.
- $a_n = \frac{1}{n}$ is strictly decreasing.
- $a_n = 5$ (constant sequence) is both increasing and decreasing (but not strictly).

Why Monotonicity Matters)

Monotonicity is a key idea in sequence problems, particularly when combined with other properties like boundedness :

- It helps in proving inequalities and estimating bounds.
- A monotonic and bounded sequence is easier to analyze and often stabilizes, especially if its values are integers.
- It supports inductive arguments, particularly when showing that terms move consistently toward or away from a target.
- It is useful for identifying extremal terms in finite sequences or proving that certain values cannot be exceeded.

1.3 Boundedness of a Sequence

A sequence is said to be **bounded** if its terms do not grow arbitrarily large or small. That is, the values of the sequence remain within some fixed range.

Definitions

Let $\{a_n\}$ be a sequence.

- **Bounded Above:** There exists a real number M such that $a_n \leq M$ for all n. *(We say M is an upper bound.)*
- **Bounded Below:** There exists a real number m such that $a_n \geq m$ for all n. *(We say m is a lower bound.)*
- **Bounded Sequence:** A sequence is bounded if it is both bounded above and bounded below. That is, there exist real numbers m and M such that

$$m \leq a_n \leq M \quad \text{for all } n.$$

Examples

- $a_n = (-1)^n$ is bounded between -1 and 1.
- $a_n = n$ is bounded below (by 1), but not above.
- $a_n = \frac{1}{n}$ is bounded between 0 and 1.
- $a_n = n^2$ is unbounded above.

Why Boundedness Matters

Boundedness plays a crucial role in controlling the behavior of sequences and proving termination or stabilization:

- It is often used to show that a sequence cannot grow indefinitely, especially when combined with monotonicity or invariants.
- In recursive problems, boundedness helps prove that values remain within a specific range or that a process must eventually stop.
- For integer sequences, boundedness can imply that only finitely many values are possible—leading to eventual repetition or constancy.
- It is useful in bounding partial sums or identifying extremal behavior in finite or constrained settings.

1.4 Periodicity of a Sequence

A sequence is said to be **periodic** if its values repeat at regular intervals. Periodicity is often seen in sequences defined using modular arithmetic, recurrence relations, or finite states.

Definition

A sequence $\{a_n\}$ is **periodic** with period $T \in \mathbb{N}$ (where $T \geq 1$) if

$$a_{n+T} = a_n \quad \text{for all } n \in \mathbb{N}.$$

The smallest such T is called the **minimal period** of the sequence.

Eventually Periodic Sequences

A sequence is **eventually periodic** if there exists an index N and a period T such that

$$a_{n+T} = a_n \quad \text{for all } n \geq N.$$

These are common in problems involving recurrence relations with finite memory or modulo constraints.

Examples

- $a_n = (-1)^n$: Periodic with period $T = 2$.
- $a_n = n \mod 3$: Periodic with period $T = 3$.
- Let $a_1 = 1$, $a_{n+1} = (a_n + 3) \mod 7$. Then $\{a_n\}$ is eventually periodic (in fact, purely periodic).
- $a_n = \lfloor \sqrt{n} \rfloor$: Not periodic, nor eventually periodic.

Why Periodicity Matters

Periodicity is a powerful concept that reduces infinite behavior to finite structure, especially in modular or recursive settings:

- It allows complex or recursive sequences to be simplified by identifying repeating patterns or cycles.
- Periodicity is common in number theory and combinatorics, particularly in problems involving modular arithmetic or finite state transitions.
- It is often proven using the pigeonhole principle—especially in bounded or discrete systems—leading to eventual repetition.
- Recognizing a cycle enables fast computation of distant terms (e.g., finding a_{2025} after identifying a period).

1.5 Arithmetic Sequences

An **arithmetic sequence** is a sequence in which each term is obtained from the previous one by adding a fixed number, called the **common difference**.

Definition

A sequence $\{a_n\}$ is arithmetic if there exists a constant $d \in \mathbb{R}$ such that

$$a_{n+1} = a_n + d \quad \text{for all } n \in \mathbb{N}.$$

In this case, d is the common difference, and the sequence takes the form:

$$a_n = a_1 + (n-1)d.$$

Examples

- $3, 6, 9, 12, \ldots$ with $a_1 = 3,\ d = 3$
- $10, 7, 4, 1, \ldots$ with $a_1 = 10,\ d = -3$
- $5, 5, 5, 5, \ldots$ is a constant sequence with $d = 0$

Useful Properties

- The difference between any two terms is proportional to their index difference:

$$a_m - a_n = (m-n)d$$

- The average of any two terms equally spaced from the beginning and end is the same as the middle term:

$$a_k = \frac{a_{k-r} + a_{k+r}}{2}$$

- The sum of the first n terms:

$$S_n = \frac{n}{2}(2a_1 + (n-1)d) = \frac{n}{2}(a_1 + a_n)$$

Why Arithmetic Sequences Matter

Arithmetic sequences frequently appear in Olympiad problems, particularly those involving number theory, symmetry, or pattern recognition:

- They often model situations with constant step changes, making them ideal for solving integer and recursive problems.
- Recognizing an arithmetic pattern allows quick computation of terms, sums, or bounds.
- Properties like constant differences and midpoint averaging are useful for bounding, estimating, and simplifying expressions.
- Olympiad problems may require proving that certain terms lie in an arithmetic progression, or exploiting the structure to work backward or forward.

1.6 Geometric Sequences

A **geometric sequence** is a sequence in which each term is obtained by multiplying the previous term by a fixed non-zero number, called the **common ratio**.

Definition

A sequence $\{a_n\}$ is geometric if there exists a constant $r \in \mathbb{R} \setminus \{0\}$ such that

$$a_{n+1} = a_n \cdot r \quad \text{for all } n \in \mathbb{N}.$$

The general formula for the n-th term is:

$$a_n = a_1 \cdot r^{n-1}.$$

Examples

- $2, 4, 8, 16, \ldots$ with $a_1 = 2$, $r = 2$
- $81, 27, 9, 3, \ldots$ with $a_1 = 81$, $r = \frac{1}{3}$
- $5, -5, 5, -5, \ldots$ with $a_1 = 5$, $r = -1$

Useful Properties

- The ratio between consecutive terms is always constant:

$$\frac{a_{n+1}}{a_n} = r$$

- The product of any two terms equidistant from the beginning and end is constant:

$$a_k^2 = a_{k-r} \cdot a_{k+r}$$

- The product of the first n terms:

$$P_n = a_1^n \cdot r^{\frac{n(n-1)}{2}}$$

- The sum of the first n terms (for $r \neq 1$):

$$S_n = a_1 \cdot \frac{1 - r^n}{1 - r}$$

Why Geometric Sequences Matter

Geometric sequences are common in Olympiad problems involving multiplication, powers, or exponential behavior:

- They naturally arise in problems involving repeated multiplication, exponential growth, or ratios between terms.

- Recognizing a geometric structure can simplify calculations, reveal closed-form expressions, or help detect recurrence patterns.
- They are often combined with other sequences or used as components in recursive constructions.
- Their properties are especially useful in problems involving symmetry, powers of integers, and multiplicative identities.

ALMO 2024 Senior P5

Define the sequences $(a_n), (b_n)$ by

$$a_n, b_n > 0, \forall n \in \mathbb{N}_+$$
$$a_{n+1} = a_n - \frac{1}{1 + \sum_{i=1}^{n} \frac{1}{a_i}}$$
$$b_{n+1} = b_n + \frac{1}{1 + \sum_{i=1}^{n} \frac{1}{b_i}}$$

1) If $a_{100}b_{100} = a_{101}b_{101}$, find the value of $a_1 - b_1$;

$a_{n+1}(1 + \sum\limits_{i=1}^{n} \frac{1}{a_i}) = a_n(1 + \sum\limits_{i=1}^{n} \frac{1}{a_i}) - 1 = a_n(1 + \sum\limits_{i=1}^{n-1} \frac{1}{a_i}) = \cdots = a_1$ So $\frac{a_1}{a_{n+2}} = 1 + \sum\limits_{i=1}^{n+1} \frac{1}{a_i} = 1 + \sum\limits_{i=1}^{n} \frac{1}{a_i} + \frac{1}{a_{n+1}} = \frac{a_1+1}{a_{n+1}}$ Thus $a_{n+2} = \dfrac{a_1}{a_1+1} a_{n+1}$ Which gives

$$\boxed{a_n = \left(\frac{a_1}{a_1+1}\right)^n (a_1+1)}$$

Similarly we have $b_n = \dfrac{b_1 + 2n - 2}{b_1 + 2n - 3} b_{n-1}$

$\frac{a_1}{a_1+1} = \frac{a_{101}}{a_{100}} = \frac{b_{100}}{b_{101}} = \frac{b_1+199}{b_1+200} \Longrightarrow a_1 - b_1 = 199$

Chapter 2 : Techniques

2.1 Induction

Well its no secret that induction is one of the main types of proofs , especially when integers and involved , and guess what all sequences have in common ? the domain is always a subset of integers !

Types of Induction:

Any way to fill the entire set you're working with is a type of induction , common ways to go about induction are :

- Forward induction $P(n) \implies P(n+1)$
- Backward induction $P(n) \implies P(n-1)$
- Cauchy's induction $P(n) \implies P(2n)$ and $P(n) \implies P(n-1)$ (the one used AM-GM)
- Strong induction $P(1) \wedge P(2) \wedge ... \wedge P(n) \implies P(n+1)$

Induction in action :

Note : the problem doesnt fully rely on induction , but its a key part of it !

Example Problem for Induction

For each integer $a_0 > 1$, define the sequence $a_0, a_1, a_2, \ldots$ for $n \geq 0$ as

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise.} \end{cases}$$

Determine all values of a_0 such that there exists a number A such that $a_n = A$ for infinitely many values of n.

Solution Walkthrough :

Understanding the Problem

The problem seems unusual at first glance, so it's helpful to clarify what it's really asking. If a value A appears infinitely many times in the sequence, then the value that follows it must also repeat. Continuing this logic, we realize that the sequence becomes **eventually periodic** — this is what the problem is actually testing.

To explore when this happens, we apply the *small cases technique*:

- $a_0 = 2$:

$$2 \to 5 \to 8 \to 11 \to 14 \to \cdots$$

- $a_0 = 3$:

$$3 \to 6 \to 9 \to 3 \to \cdots$$

- $a_0 = 4$:

$$4 \to 2 \to \cdots$$

- $a_0 = 7$:

$$7 \to 10 \to 13 \to 16 \to 4 \to 2 \to \cdots$$

From this casework, we observe that values divisible by 3 (i.e., $a_0 \equiv 0 \pmod 3$) eventually form a cycle, while those not divisible by 3 do not. This motivates us to analyze the sequence modulo 3.

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Case 1: $a_0 \equiv 2 \pmod 3$

To apply the square root step, the term must be a perfect square. However, observe that:

$$m^2 \equiv 0 \text{ or } 1 \pmod 3 \quad \text{for all } m \in \mathbb{Z}.$$

Thus, if $a_n \equiv 2 \pmod 3$, it cannot be a perfect square, and the square root step will never occur. So, the sequence will always evolve via the rule:

$$a_{n+1} = a_n + 3.$$

Hence, it is strictly increasing and cannot be periodic.

Inductive proof:

- **Base case:** $a_0 \equiv 2 \pmod 3$ — clearly true.
- **Inductive step:** Suppose $a_n \equiv 2 \pmod 3$. Since it cannot be a perfect square, we have:

$$a_{n+1} = a_n + 3 \equiv 2 \pmod 3.$$

  The induction step holds.

Therefore, by induction, all terms satisfy $a_n \equiv 2 \pmod 3$, and the sequence is strictly increasing and non-periodic.

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Case 2: $a_0 \equiv 0 \pmod 3$

Proof idea: Strong Induction. We claim that all such sequences eventually fall into the cycle $3 \to 6 \to 9 \to 3 \to \cdots$.

Base case: Verified directly.

Inductive step: Assume that all values $3, 6, 9, \ldots, 3n$ eventually fall into the cycle. We show that $3n + 3$ does too.

- **Case 1:** If $3n + 3$ is a perfect square, the square root is a multiple of 3 and lies in the base chain. So it enters the cycle.
- **Case 2:** If not a perfect square, let $m \in \mathbb{N}$ be such that:

$$3n + 3 < (3m)^2.$$

  Then eventually, the sequence will hit $(3m)^2$ after adding 3 repeatedly. Since $3m$ is a multiple of 3, $(3m)^2$ maps via the square root step to $3m$, which lies in the already-proven cycle.

Thus, by strong induction, all $a_0 \equiv 0 \pmod 3$ lead to eventual periodicity.

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Case 3: $a_0 \equiv 1 \pmod 3$

Observation: In this case, the sequence always reaches a number $\equiv 2 \pmod 3$, which, as we've already shown, causes the sequence to increase forever without becoming periodic.

Inductive Argument:

Let us prove that every sequence starting with $a_0 = 3k + 1$ eventually hits a number congruent to $2 \pmod 3$.

Inductive Hypothesis: For all $k < n+1$, if $a_0 = 3k+1$, then the sequence eventually reaches a term $\equiv 2 \pmod 3$.

Inductive Step: Let $a_0 = 3(n+1)+1$. Let k^2 be the first perfect square the sequence reaches. Since the sequence increases by 3 at each non-square step, and perfect squares grow faster than linear, eventually a_n becomes a perfect square. Note:

$$k^2 \leq (a_0 - 3)^2 < a_0 + 3,$$

so $k \leq a_0 - 3$. Moreover, $k \equiv 1 \text{ or } 2 \pmod 3$.

If $k \equiv 2 \pmod 3$, then the resulting square will lead to a term $\equiv 2 \pmod 3$, and we are done.

If $k \equiv 1 \pmod 3$, then by inductive hypothesis, the square root step brings us to a case that eventually reaches $\equiv 2 \pmod 3$. Either way, the sequence becomes strictly increasing and non-periodic.

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Final Conclusion

Putting all cases together: - If $a_0 \equiv 0 \pmod 3$, the sequence becomes periodic. - If $a_0 \equiv 1 \text{ or } 2 \pmod 3$, the sequence is non-periodic.

$$\boxed{\text{The sequence is eventually periodic if and only if } 3 \mid a_0.}$$

Takeaway

Induction is a powerful tool. The hardest part is knowing what to hit with the inductive hammer.

Problem source : IMO 2017 P1

2.2 Bounding Techniques

One powerful method in Olympiad problems involves exploiting whether a sequence is *bounded* or *unbounded*. You can also compare growth rates using inequalities or even asymptotic behavior.

A Cool Theorem

Let $P(x)$ and $Q(x)$ be polynomials with leading terms px^n and qx^m, respectively.

- If $n > m$, then $P(x)$ eventually outgrows $Q(x)$.
- If $n < m$, then $Q(x)$ eventually outgrows $P(x)$.
- If $n = m$, then:
  - If $p > q$, then $P(x) > Q(x)$ for large x,
  - If $p < q$, then $Q(x) > P(x)$,
  - If $p = q$, then $P(x) \sim Q(x)$ asymptotically.

Bounding in Action

Example Problem – Bounded Sequence

Does there exist a sequence of positive real numbers $\{a_i\}_{i=1}^{\infty}$ satisfying:

$$\sum_{i=1}^{n} a_i \geq n^2 \quad \text{and} \quad \sum_{i=1}^{n} a_i^2 \leq n^3 + 2025n$$

for all positive integers n?

This problem clearly calls for **bounding**, so a natural first step is to try a known inequality — specifically, the Cauchy–Schwarz inequality. Applying it yields:

$$(\sum_{i=1}^{n} a_i)^2 \leq n. \sum_{i=1}^{n} a_i^2 = n^4 + 2025n^2 \leq (n^2 + 2025)^2$$

Taking square roots:

$$\sum_{i=1}^{n} a_i \leq n^2 + 2025.$$

Now, let us define:

$$S_n = \sum_{i=1}^{n} a_i.$$

Then we have:

$$a_n = S_n - S_{n-1} \geq n^2 - ((n-1)^2 + 2025) = 2n - 2026.$$

The kill move: Suppose the sequence does exist. Then:

$$\sum_{i=1}^{n} a_i^2 \le n^3 + 2025n.$$

Now, let's consider the tail of the sequence starting at a large enough index to use well known identities — say, $i = 1013$. Then:

$$\sum_{i=1013}^{n} a_i^2 \le \sum_{i=1}^{n} a_i^2 \le n^3 + 2025n.$$

But from the earlier bound:

$$a_i \ge 2i - 2026 \quad \text{for all } i \ge 1013.$$

Thus,

$$\sum_{i=1013}^{n} a_i^2 \ge \sum_{i=1013}^{n} (2i - 2026)^2.$$

Now make a substitution: let $j = i - 1013$. Then $i = j + 2013$, and the sum becomes:

$$\sum_{j=0}^{n-1013} (2j)^2 = 4 \sum_{j=0}^{n-1013} j^2 = 4 \cdot \frac{(n-1013)(n-1012)(2n-2015)}{6}.$$

This is a degree-3 polynomial in n with leading coefficient $\frac{4}{3}$, while the upper bound $n^3 + 2025n$ has leading coefficient 1. For large n, the lower bound exceeds the upper bound — a contradiction.

Conclusion: No such sequence can exist.

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What We Learn

Bounding can be extremely powerful, especially when combined with inequalities and asymptotic behavior. In this problem, we showed that the lower bound grows faster than the upper bound — something you can only see clearly by comparing leading terms of polynomials. The contradiction proves the sequence is impossible.

Problem source Turkey EGMO TST P2 2025

Example Problem – unbounded Sequence usefulness

For any natural number n and positive integer k, we say that n is $k - good$ if there exist non-negative integers $a_1, \ldots, a_k$ such that

$$n = a_1^2 + a_2^4 + a_3^8 + \ldots + a_k^{2^k}.$$

Is there a positive integer k for which every natural number is $k - good$?

In this kind of problems you either find a construction or find a contradiction ! , it turns out contradiction is the way to go !

Assume, for the sake of contradiction, that such a k exists.

Let N be a large positive integer. Then every integer in the set $\{1, 2, \ldots, N\}$ must be expressible in the given form.

Each term $a_i^{2^i}$ in the sum must satisfy:

$$a_i^{2^i} \leq N \quad \Rightarrow \quad a_i \leq N^{1/2^i}$$

Therefore:

- For $a_1^2 \leq N$: there are at most $\sqrt{N} + 1$ possible values.
- For $a_2^4 \leq N$: at most $\sqrt[4]{N} + 1$ values.
- For $a_3^8 \leq N$: at most $\sqrt[8]{N} + 1$ values.
- ...
- For $a_k^{2^k} \leq N$: at most $\sqrt[2^k]{N} + 1$ values.

The total number of combinations (i.e., tuples $(a_1, \ldots, a_k)$) is at most:

$$\prod_{j=1}^{k} \left(N^{1/2^j} + 1 \right)$$

This is an upper bound on how many distinct integers can be represented using the given expression.

Since all integers from 1 to N must be representable, we must have:

$$\prod_{j=1}^{k} \left(N^{1/2^j} + 1 \right) \geq N$$

Now consider the growth of this product. The dominant term is:

$$\prod_{j=1}^{k} N^{1/2^j} = N^{\sum_{j=1}^{k} \frac{1}{2^j}} = N^{1 - \frac{1}{2^k}}$$

So the total number of representable values is at most:

$$N^{1 - \frac{1}{2^k}} \cdot (\text{lower order terms})$$

This function grows **strictly slower** than N as $N \to \infty$.

This contradicts the assumption that *every* number up to N must be representable, since:

$$N^{1-\frac{1}{2^k}} < N \quad \text{for all } k \geq 1$$

Hence, the number of available representations is eventually too small to cover all of $\{1, 2, \ldots, N\}$ as N grows.

Conclusion

Our assumption leads to a contradiction. Therefore, such a k does not exist.

$$\boxed{\text{No such } k \text{ exists.}}$$

Key Take Away : sometimes taking something large enough is very useful

Problem source : Estonia IMO TST P2 2023

2.3 Working with Special Terms

Example Problem

There are $n \geq 3$ positive real numbers $a_1, a_2, \ldots, a_n$. For each $1 \leq i \leq n$ we let $b_i = \frac{a_{i-1}+a_{i+1}}{a_i}$ (here we define a_0 to be a_n and a_{n+1} to be a_1). Assume that for all i and j in the range 1 to n, we have $a_i \leq a_j$ if and only if $b_i \leq b_j$. Prove that $a_1 = a_2 = \cdots = a_n$.

The entire problem boils down to considering the minimum and maximum terms of the sequence (there are other solutions that dont use this idea but are way longer) , the solution goes as follows :
Let k and ℓ be indices such that $a_k = \min(a_1, \ldots, a_n)$ and $a_\ell = \max(a_1, \ldots, a_\ell)$. Then

$$b_k = \frac{a_{k-1} + a_{k+1}}{a_k} \geq \frac{a_k + a_k}{a_k} = 2,$$

and similarly

$$b_\ell = \frac{a_{\ell-1} + a_{\ell+1}}{a_\ell} \leq \frac{a_\ell + a_\ell}{a_\ell} = 2.$$

But since $a_k \leq a_i \leq a_\ell$ for all i, we must have $b_k \leq b_i \leq b_\ell$ for all i, so $b_i = 2$ for all i. Notice $b_k = b_l \implies a_k = a_l \implies a_1 = a_2 = \ldots = a_n$.

Problem source : EGMO 2023 P1

Example Problem

Determine all composite integers $n > 1$ that satisfy the following property: if d_1, d_2, $\ldots$, d_k are all the positive divisors of n with $1 = d_1 < d_2 < \cdots < d_k = n$, then d_i divides $d_{i+1} + d_{i+2}$ for every $1 \leq i \leq k-2$.

The key idea is to consider the first few devisors and the last few devisors and observe a pattern , here is the solution outline :

From $d_{k-2} \mid d_{k-1} + d_k$ one can get that $d_{k-2} \mid d_{k-1}$ since $d_{k-2} \mid n = d_k$ Then $d_2 = \frac{n}{d_{k-1}} \mid \frac{n}{d_{k-2}} = d_3$ so $d_2 \mid d_3$. Here, since $d_2 \mid d_3 + d_4$, we get that $d_2 \mid d_4$. Again, we can easily show $d_{k-3} | d_{k-1}$ in the same way, which leads to $d_{k-3} \mid d_{k-2}$

By induction, $d_1 \mid d_2 \mid \cdots \mid d_k$

If $p \mid n$ and $q \mid n$ for prime $p > q$ $\frac{n}{p} \mid \frac{n}{q}$ so $q \mid p$ and this is contradiction

Hence, $n = p^{k-1}$ This indeed fits the condition

Alternative solution : Assume $p < q$ be the smallest two prime divisors of n. Then $d_{k-1} = \frac{n}{p}$. Assume $d_m = \frac{n}{q}$ for some m, and $d_{m+1} = \frac{n}{p^{c+1}}$ and $d_{m+2} = \frac{n}{p^c}$ for some nonnegative integer c. Then, since $d_m \mid d_{m+1} + d_{m+2}$, $p^{c+1} \mid q + pq$ and $p|q$ which is a contradiction. Therefore n does not have two distinct prime factors; n=p^t for some prime p and a positive integer $t \neq 1$. It's easy to show that this suffices.

Problem source : IMO 2023 P1

2.4 Algorithms

Long story short we will consider algorithms as a deterministic process that repeats over and over , we saw it previously in the IMO 2023 P1 Solution 1 as it relied on an algorithm to shrink down the devisors that dont do $d_i|d_{i+1}$ into nothingness , we will also see it in the following problem :

Example Problem

The positive integers $a_0, a_1, a_2, \ldots, a_{3030}$ satisfy

$$2a_{n+2} = a_{n+1} + 4a_n \text{ for } n = 0, 1, 2, \ldots, 3028.$$

Prove that at least one of the numbers $a_0, a_1, a_2, \ldots, a_{3030}$ is divisible by 2^{2020}.

Suppose that for some $1 \le k \le 3029$, we have a_k is odd. Then, we get that

$$a_{k+1} = \frac{a_k}{2} + 2a_{k-1}$$

so thus we have that a_k is even in these cases. Now, consider $1 \le k \le 3028$ and suppose a_k is $2 \pmod 4$. Then, we can get that

$$a_{k+1} = \frac{a_k}{2} + 2a_{k-1}$$

which is odd, a contradiction. Thus, for all $1 \le k \le 3028$, we have $4 \mid a_k$.

Now, consider the sequence $\{b_i\}_{i=0}^{3027}$ satisfying

$$b_i = \frac{a_{i+1}}{4}$$

This satisfies the same recurrence, so if we show that $2^k \mid b_i$ for some i, we get $2^{k+2} \mid a_{i+1}$. Then, the problem is very easily trivialized by repeating this algorithm 1010 times (in all) so that we get some c_0. Then, we get that we can iterate backwards to get $2^{2020} \mid a_k$ for some k.

Takeaway : sometimes simplification can be repeated and that can be useful !

Problem source EGMO 2020 P1

2.5 Defining New Sequences

Sometimes you have to manipulate the sequence too much , this can become easier when you define a new sequence related to the original sequence, observe the following problem:

Example Problem

Consider two infinite sequences $a_0, a_1, a_2, \dots$ and $b_0, b_1, b_2, \dots$ of real numbers such that $a_0 = 0,\ b_0 = 0$ and

$$a_{k+1} = b_k, \quad b_{k+1} = \frac{a_k b_k + a_k + 1}{b_k + 1}$$

for each integer $k \geq 0$. Prove that $a_{2024} + b_{2024} \geq 88$

Idea : recall $(a_k+1)(b_k+1) = a_kb_k + a_k + b_k + 1$ which is kinda similar to the numorator lets exploit that !

Solution : It is easy to show that $(a_{k+1}+1)(b_{k+1}+1) = (a_k+1)(b_k+1)+1$. Hence if we define the sequences $A_k = a_k + 1$ and $B_k = b_k + 1$, then $A_{k+1}B_{k+1} = A_kB_k + 1$. Since $A_0 = a_0 + 1 = 1$ and $B_0 = b_0 + 1 = 1$, we have $A_0B_0 = 1 \cdot 1 = 1$. The forementioned gives $A_{2024}B_{2024} = 1 + 2024 = 2025$. Therefore using AM-GM we have

$$(a_{2024}+1) + (b_{2024}+1) = A_{2024} + B_{2024} \geq 2\sqrt{A_{2024} \cdot B_{2024}} = 2\sqrt{2025} = 90.$$

Hence $a_{2024} + b_{2024} \geq 88$ as desired.

Problem source : MEMO 2024 P1

Example Problem

Let $x_1, x_2, \dots, x_{2023}$ be pairwise different positive real numbers such that

$$a_n = \sqrt{(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)}$$

is an integer for every $n = 1, 2, \dots, 2023$. Prove that $a_{2023} \geqslant 3034$.

Define $A_n := x_1 + x_2 + \cdots + x_n$ and $B_n := \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}$. Observe that

$$\begin{aligned} a_{n+1}^2 &= (A_n + x_{n+1})\left(B_n + \frac{1}{x_{n+1}}\right) \\ &= A_nB_n + \frac{A_n}{x_{n+1}} + B_nx_{n+1} + 1 \\ &\geq A_nB_n + 2\sqrt{A_nB_n} + 1 = (a_n+1)^2. \end{aligned}$$

Hence $a_{n+1} \geq a_n + 1$.

We now prove the stronger inequality $a_{n+2} \geq a_n + 3$. Suppose, by way of contradiction, that $a_{n+2} = a_n + 2$. Then equality cases in both inequalities above (i.e. for a_{n+1} and a_{n+2})

hold, so $x_{n+1} = \sqrt{\frac{A_n}{B_n}}$ and

$$x_{n+2}^2 = \frac{A_n + x_{n+1}}{B_n + \frac{1}{x_{n+1}}} = \frac{A_n + \sqrt{\frac{A_n}{B_n}}}{B_n + \sqrt{\frac{B_n}{A_n}}} = \frac{A_n}{B_n}.$$

(The last equality is true because $\frac{x_{n+1}}{\frac{1}{x_{n+1}}} = x_{n+1}^2 = \frac{A_n}{B_n}$.) This contradicts the fact that the x_j are pairwise distinct. Therefore $a_{n+2} \geq a_n + 3$. Since $a_1 = 1$, $a_{2023} \geq a_{2021} + 3 \geq \cdots \geq a_1 + 3033 = 3034$.

Problem source : IMO 2023 P4

Chapter 3 : Practice Problems

ISL 2006 A1

A sequence of real numbers $a_0,\ a_1,\ a_2, \ldots$ is defined by the formula

$$a_{i+1} = \lfloor a_i \rfloor \cdot \langle a_i \rangle \qquad \text{for} \quad i \geq 0;$$

here a_0 is an arbitrary real number, $\lfloor a_i \rfloor$ denotes the greatest integer not exceeding a_i, and $\langle a_i \rangle = a_i - \lfloor a_i \rfloor$. Prove that $a_i = a_{i+2}$ for i sufficiently large.

ISL 2015 A1

Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \geq \frac{k a_k}{a_k^2 + (k-1)}$$

for every positive integer k. Prove that $a_1 + a_2 + \ldots + a_n \geq n$ for every $n \geq 2$.

IMO 2014 P1

Let $a_0 < a_1 < a_2 < \ldots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that

$$a_n < \frac{a_0 + a_1 + a_2 + \cdots + a_n}{n} \leq a_{n+1}.$$

APMO 2019 P2

Let m be a fixed positive integer. The infinite sequence $\{a_n\}_{n\geq 1}$ is defined in the following way: a_1 is a positive integer, and for every integer $n \geq 1$ we have

$$a_{n+1} = \begin{cases} a_n^2 + 2^m & \text{if } a_n < 2^m \\ a_n/2 & \text{if } a_n \geq 2^m \end{cases}$$

For each m, determine all possible values of a_1 such that every term in the sequence is an integer.

ISL 2022 A1

Let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers with the property that

$$(a_{n+1})^2 + a_n a_{n+2} \leq a_n + a_{n+2}$$

for all positive integers n. Show that $a_{2022} \leq 1$.